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Magnetohydrodynamic self-consistent exact helical solutions

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Abstract

We consider the idealized case of a one-component plasma with aligned fluid velocity and current density. Constant density and pressure as well as zero external magnetic field are also assumed. We show that suitably determined axially symmetric helical current densities within a straight infinite cylinder are exact self-consistent solutions of magnetohydrodynamics. Self-consistent here means that the magnetic field is the field produced by the current density itself. The equation of motion gives a nonlinear differential equation that relates the axial $v_z(\rho)$ and the azimuthal $v_\varphi(\rho)$ velocities as functions of radial distance ρ. Prescribing one of these gives a specific solution for the other. The solutions can be understood as a set of helix-shaped charged particle trajectories that spiral self-consistently through the magnetic field that they themselves give rise to. Four different specific exact solutions are given: (i) a single particle outside a rectilinear line current, (ii) current on a thin cylinder, (iii) current density with constant angular velocity and (iv) current density with constant axial velocity, both within a cylinder of finite radius.

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1. Introduction

A starting point in plasma theory [1–3] is frequently the study of charged particle motion in given external magnetic fields. A recent example of this is a study by Yafaev [4] on the motion of a charged particle in the magnetic field from an infinite rectilinear line current. However, when there are many moving particles involved they produce a magnetic field themselves. For small systems this is often a negligible effect, but since the magnetic field, just like the electric field, is a long-range field which, unlike the electric field, is not screened on the Debye length scale, self-fields inevitably become important for larger systems. The nonlinear effects of these self-fields can be taken into account in magnetohydrodynamics (MHD) [1–3, 5, 6] and here we will be interested in solutions with zero external field. We will call such solutions describing the motion of charged particles in the magnetic field that they themselves produce *self-consistent*.

These difficult nonlinear problems must normally be approached numerically. Problems with *cylindrical symmetry* are, however, more likely to yield to analytical methods since they become effectively two dimensional, and we will confine our attention to them. There is extensive literature on such cylindrically symmetric *helical* flows [6–12]. So far, however, most workers have neglected the inertia of the particles and either assumed, so-called plasma equilibrium, $\boldsymbol{j} \times \boldsymbol{B}/c = \nabla p$ [6, 7] or the force-free case, $\boldsymbol{j} \times \boldsymbol{B} = \boldsymbol{0}$ [1, 5, 13–15].

There is also considerable experimental [16–19] and theoretical [20, 21] evidence that flux tubes and helical current patterns form spontaneously in real plasmas. The stabilization may be due to the existence of a conserved circulation theorem for plasmas [22]. Current structures that minimize magnetic energy can be shown to be consistent with plasma thermal equilibrium [23–25]. In view of all this the simple and fundamental, but highly idealized, new solutions given below should be of some interest.

We will show that certain axially symmetric helical current densities, within a cylinder, are exact solutions of ideal magnetohydrodynamics for a homogeneous plasma with constant pressure and density as well as zero external field. It is also assumed that the fluid velocity and the current density are *aligned*. The paper is organized as follows. In section 2 we discuss the motion of a charged particle outside a line current. While trivial from the self-consistent point of view this section serves to introduce notation and several concepts such as helical trajectories and their pitch angle, the Lambert *W*-function, the dimensionless magnetic number ν etc, that will appear again in subsequent sections. In section 3 we then present our general assumptions about the current density and the magnetic field that is produced. After this (in section 4) the special case of current only on a thin cylindrical surface is studied. For this case, the equation of motion reduces to an algebraic equation and the solution depends only on ν. The reader is now prepared for the more difficult case of current density within a cylinder coming next. In section 5 one finds in the general case a *class* of solutions. In the two special cases of constant angular velocity and constant axial velocity (section 6) explicit analytical solutions are given.

2. Particle outside line current

First some notation and kinematics. We will use cylindrical coordinates, ρ, φ, z defined through, $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, $z = z$, in terms of Cartesian coordinates. We also use the corresponding unit vectors $\boldsymbol{e}_\rho(\varphi)$ and $\boldsymbol{e}_\varphi(\varphi)$ time derivatives of which obey the relations $\dot{\boldsymbol{e}}_\rho = \dot\varphi\boldsymbol{e}_\varphi$, $\dot{\boldsymbol{e}}_\varphi = -\dot\varphi\boldsymbol{e}_\rho$. A position vector is given by $\boldsymbol{r} = \rho\boldsymbol{e}_\rho + z\boldsymbol{e}_z$, and the corresponding velocity is $\boldsymbol{v} = \dot\rho\boldsymbol{e}_\rho + \rho\dot\varphi\boldsymbol{e}_\varphi + \dot z\boldsymbol{e}_z$, so that $v_\rho = \dot\rho$ and $v_\varphi = \rho\dot\varphi$. The equation for a helix of radius $\rho = R$ can now be written ($\dot\varphi = \omega = \text{const}$) as

$$\boldsymbol{r}(t) = R\boldsymbol{e}_\rho(\omega t) + v_z t\boldsymbol{e}_z, \tag{1}$$

where we use time t as parameter. Introducing the *pitch angle* θ through

$$v_\varphi = R\omega = v\sin\theta, \qquad v_z = v\cos\theta, \tag{2}$$

or equivalently, $v_\varphi/v_z = \tan\theta$, one gets

$$\boldsymbol{v}(t) = v[\sin\theta\,\boldsymbol{e}_\varphi(\omega t) + \cos\theta\,\boldsymbol{e}_z], \tag{3}$$

for the tangent (velocity) vector of this helix.

The motion of a charged particle outside an infinite rectilinear current in a thin wire has been solved, for both the classical and the quantum case, by Yafaev [4]. The main feature of these solutions is that the net overall motion of the particle (if it moves at all) is a drift parallel to the current, if positively charged, otherwise in the opposite direction. From this one draws the general conclusion that such a wire current induces a parallel current among charged particles outside it.

Here we will only consider one class of possible motions of the particle, the helical motion, but we start from the general equations of motion. These can be found from the Lagrangian

$$L = \frac{1}{2}m\boldsymbol{v}^2 + \frac{q}{c}\boldsymbol{A}\cdot\boldsymbol{v} \tag{4}$$

where m is mass, q electric charge, and $\boldsymbol{A}$ the vector potential. For a line current I along the z-axis one finds that

$$\boldsymbol{A} = -\frac{2I}{c}\ln\rho\,\boldsymbol{e}_z, \tag{5}$$

using cylindrical coordinates. The corresponding magnetic field is $\boldsymbol{B}(\boldsymbol{r}) = \frac{2I}{c\rho}\boldsymbol{e}_\varphi$. Thus,

$$L(\rho,\dot\rho,\dot\varphi,\dot z) = \frac{1}{2}m(\dot\rho^2+\rho^2\dot\varphi^2+\dot z^2) - \frac{2qI}{c^2}\dot z\ln\rho, \tag{6}$$

and, since the coordinates φ and z are cyclic one finds the constants of motion,

$$m\rho^2\dot\varphi = L_z, \tag{7}$$

$$m\dot z - \frac{2qI}{c^2}\ln\rho = p_z. \tag{8}$$

With these the ρ-equation of motion, $m\ddot\rho - m\rho\dot\varphi^2 + (2qI/c^2)(\dot z/\rho) = 0$, can be written as

$$m\ddot\rho = \frac{L_z^2}{m\rho^3} - \frac{p_c}{m\rho}(p_z + p_c\ln\rho), \tag{9}$$

where we have introduced

$$p_c \equiv \frac{2qI}{c^2}. \tag{10}$$

It is clear from this that for positive p_c there will be stable solutions with $\ddot\rho = \dot\rho = 0$, and $\rho =$ constant, corresponding to helical trajectories.

The radius $R = \rho$ of these trajectories can be found by putting the left-hand side of (9) equal to zero and solving for ρ. Maple [26, 27] gives

$$R = \exp\left\{\frac{1}{2}W\left[2\left(\frac{L_z}{p_c}\right)^2\exp\left(2\frac{p_z}{p_c}\right)\right] - \frac{p_z}{p_c}\right\}, \tag{11}$$

for the radius of the helix. Here W is Lambert's W-function [28–30], the inverse of the function,

$$f(W) = W\exp(W). \tag{12}$$

It can be regarded as a less well-known elementary function and will appear again below.

For the helical trajectory we must have $L_z = mR^2\omega$, so that the angular velocity $\dot\varphi = \omega$ must also be constant. Use of this in the ρ-equation of motion, with $\ddot\rho = 0$, gives

$$mR\omega^2 = p_c\frac{v_z}{R}, \tag{13}$$

so that $\dot{z} = v_z$ must also be constant. Comparison with equation (2) now gives

$$\sin^2\theta = \frac{p_c}{mv}\cos\theta, \tag{14}$$

for the pitch angle θ of the helical trajectory of a single particle spiralling around an infinite line current with speed v.

One can rewrite the constant p_c/m here in a more illuminating form. Putting $q = Ze$, and $I = Q/T = Ne/T = (N/\ell)(\ell/T)e = (N/\ell)v_d e$, for the line current, such that N is the number of conduction electrons in a piece of wire of length ℓ, and v_d their drift speed, we get

$$\frac{p_c}{m} = 2Zv_d\nu, \tag{15}$$

where

$$\nu \equiv \frac{Nr_e}{\ell} \qquad \text{and} \qquad r_e \equiv \frac{e^2}{mc^2}. \tag{16}$$

Here r_e is the so-called classical electron radius. ν is a dimensionless 'magnetic' number.

3. Helical current density

We now proceed from a single particle to a current density in a plasma of constant pressure. Apart from helical symmetry we also assume that we are dealing with a one-component plasma. This means that the current is assumed to be due to one kind of particle, with mass m, charge e and (constant) number density n, moving through a neutralizing background of zero net current density. Usually one then thinks of electrons but the current may very well be due to positive ions, or even composite quasi-particles, moving through a background electron density of zero net current. In any case, the main feature here is the assumption of aligned fluid velocity and current density.

In terms of cylindrical coordinates we thus take the electric current density to be

$$\boldsymbol{j}(\boldsymbol{r}) = en\,\boldsymbol{v}(\boldsymbol{r}) = en[v_\varphi(\rho)\boldsymbol{e}_\varphi(\varphi) + v_z(\rho)\boldsymbol{e}_z], \tag{17}$$

for $\rho = \sqrt{x^2+y^2} < R$, and $\boldsymbol{j} = \boldsymbol{0}$ outside this cylinder. It is well known that if $v_\varphi = 0$ there will be pinching [15, 19, 31] and resulting instability. Briefly, the idea behind the assumption of a helical current is that, for correctly chosen angular velocity, the centrifugal force will balance the pinching force. More accurately, the azimuthal component corresponds to a rotational motion whose centripetal acceleration is caused by the pinching force. The result is a circular motion with constant radius, and no pinching.

Assuming constant pressure, the magnetohydrodynamic equation of motion is [3, 6]

$$mn\left(\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right) = \frac{1}{c}\boldsymbol{j}\times\boldsymbol{B}. \tag{18}$$

Here

$$\boldsymbol{j} = en\,\boldsymbol{v} = \frac{c}{4\pi}\nabla\times\boldsymbol{B} \tag{19}$$

and we assume that $\boldsymbol{j}$ is the sole source of $\boldsymbol{B}$ (no external field) and that all para- or diamagnetic responses of the plasma can be neglected. With our current (17) this vector equation will only have one component. A simple calculation gives

$$-\frac{v_\varphi^2(\rho)}{\rho} = \frac{e}{mc}[v_\varphi(\rho)B_z(\rho) - v_z(\rho)B_\varphi(\rho)] \tag{20}$$

for the (radial) component along $\boldsymbol{e}_\rho(\varphi)$. One notes that the equilibrium MHD equations $\nabla\cdot\boldsymbol{v} = \nabla\cdot\boldsymbol{B} = 0$ and $\nabla\times(\boldsymbol{v}\times\boldsymbol{B}) = \boldsymbol{0}$ are valid in these calculations.

The magnetic field from a helical line current can be found in the literature [32, 33]. For the present case of a continuous helical current density use of equation (19) and $\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r} = (4\pi/c) \int \boldsymbol{j} \cdot \mathrm{d}\boldsymbol{S}$ allows us to express the magnetic field components simply in terms of the components of the velocity field.

One finds that

$$B_\varphi(\rho) = \frac{4\pi}{c} en \frac{1}{\rho} \int_0^\rho v_z(\rho')\rho' \, \mathrm{d}\rho', \tag{21}$$

and

$$B_z(\rho) = \frac{4\pi}{c} en \int_\rho^R v_\varphi(\rho') \, \mathrm{d}\rho'. \tag{22}$$

Note that for $\rho \geqslant R$ we must have $B_z(\rho) = 0$, since we, essentially, are outside a family of concentric infinite solenoids.

The assumption that the velocity and current density abruptly become zero at $\rho = R$ may seem unphysical. Since we are assuming ideal MHD, however, viscosity and resistivity are assumed zero. Infinite velocity gradients are thus mathematically consistent, but this is of course one of the idealizations that limit the applicability of the results obtained.

4. Current on cylinder surface

Let us first consider the case of current only on the surface of the cylinder. Formation of helical cylindrical current sheets has in fact been discovered in numerical studies [11, 34]. Such a current density also corresponds to a current in a helical coil, a problem that has been addressed by Jefimenko [35].

To study this limiting case we assume that $v_z(\rho) = v\cos\theta$ and $v_\varphi(\rho) = v\sin\theta$ for $R - \delta < \rho < R$ and zero elsewhere. Here θ is the pitch angle of the helix, see equation (2). We thus have current density on a cylindrical shell of thickness δ. Calculating the magnetic field is trivial using equations (21) and (22). Using these equation (20) becomes

$$-\frac{\sin^2\theta}{1 - \frac{1}{2}\frac{\delta}{R}} = 2\pi n r_e R\delta \left[\sin^2\theta - \frac{1 - \frac{3}{4}\frac{\delta}{R}}{1 - \frac{1}{2}\frac{\delta}{R}} \cos^2\theta \right]. \tag{23}$$

at $\rho = R - \delta/2$. We now let $\delta/R \to 0$. Since, in this case, the number density $n = N/(2\pi R\delta\ell)$, we find that the dimensionless constant

$$2\pi n r_e R\delta \to \nu = N r_e/\ell, \tag{24}$$

see equation (16). This then results in an equation,

$$-\sin^2\theta = \nu[\sin^2\theta - \cos^2\theta], \tag{25}$$

for the pitch angle.

The right-hand side here is essentially the magnetic force from the current distribution on itself, and we see that it is zero when $\sin^2\theta = \cos^2\theta$, i.e. for a pitch angle of 45°. This is in agreement with the force-free coil of Jefimenko [35], example 13-8.2. The left-hand side is due to inertia (mass times acceleration) and is needed for a self-consistent solution of freely moving charges not constrained by a coil. In this case we may rewrite (25) and obtain

$$\tan^2\theta = \frac{1}{1 + 1/\nu}, \tag{26}$$

as an alternative equation for the pitch angle of a self-consistent current density on a cylindrical shell.

5. Constant angular velocity

We now proceed to a current density that fills the cylinder ($\rho \leqslant R$). As long as the magnetic field B_z due to the angular velocity can be neglected equation (20) is solved by a constant angular velocity ω and a constant translational velocity v_z connected by

$$R\omega = \sqrt{2\nu}v_z. \tag{27}$$

When the magnetic field due to the rotation becomes more important for larger ν-values, a φ-pinch will develop, in addition to the z-pinch, and both can no longer be constant. In this section, we find the exact solution for $v_z(\rho)$ assuming constant ω, in the next we find $\omega(\rho)$ for constant v_z.

If we introduce the definitions

$$F(\rho) \equiv \int_0^\rho v_z(\rho')\rho' \, \mathrm{d}\rho' \tag{28}$$

$$G(\rho) \equiv \int_0^\rho v_\varphi(\rho') \, \mathrm{d}\rho' \tag{29}$$

we can write our equation of motion (20)

$$A\left(\frac{\mathrm{d}G}{\mathrm{d}\rho}\right)^2 + [G(R) - G(\rho)]\rho\frac{\mathrm{d}G}{\mathrm{d}\rho} = \frac{F(\rho)}{\rho}\frac{\mathrm{d}F}{\mathrm{d}\rho}. \tag{30}$$

Here,

$$A \equiv \frac{mc^2}{4\pi e^2 n}, \tag{31}$$

and the initial conditions are $F(0) = G(0) = 0$. Equation (30) gives a class of solutions; one can solve the nonlinear differential equation for F if G is specified, or vice versa.

Using the number density $n = \frac{N}{\pi R^2 \ell}$, where N/ℓ is the number of particles per length moving in the cylinder, we find that

$$A = \frac{1}{\nu}\frac{R^2}{2}, \tag{32}$$

where ν is given by equation (16). This shows that A has dimension area. The dimensionless magnetic number ν should normally be at most order of magnitude unity, otherwise one cannot neglect the response of the plasma [23]. For a cylinder of radius R we can write $\nu = n\pi R^2 r_e$. For typical ionospheric densities of $n \approx 10^{10}$ m^{-3} one finds that $\nu = 1$ requires a radius R of roughly 100 m, when r_e is the classical electron radius. If the classical proton radius is used the cylinder radius R must be 4.5 km in order for ν to reach unity. For fusion plasmas with $n \approx 10^{20}$ m^{-3} these radii reduce to 1 mm and 4.5 cm, respectively.

We first assume that the rotational motion is a rigid rotation with angular velocity ω and put $v_\varphi(\rho) = \rho\omega$. The function $G(\rho)$ is then $G(\rho) = \omega\rho^2/2$ and equation (30) gives the differential equation

$$F(\rho)\frac{\mathrm{d}F}{\mathrm{d}\rho} = \frac{(\omega R)^2}{2}\left[\frac{1}{\nu} + 1 - \left(\frac{\rho}{R}\right)^2\right]\rho^3, \tag{33}$$

for F and $\rho v_z = \mathrm{d}F/\mathrm{d}\rho$. Solving this equation is trivial. One finds that

$$v_z(\rho) = \frac{\omega R}{\sqrt{\nu}}\frac{1 + \nu[1 - (\rho/R)^2]}{\sqrt{1 + \nu[1 - (2/3)(\rho/R)^2]}}, \tag{34}$$

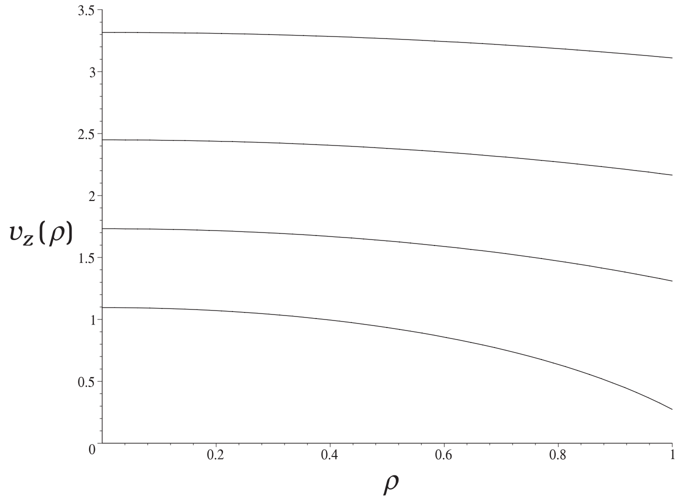

Figure 1. Plot of the function $v_z(\rho)$ of equation (34) for $\omega = R = 1$ and $\nu = 1/10, 1/5, 1/2, 5$ in order from top to bottom.

when $v_\varphi(\rho) = \rho\omega$; in this case the axial velocity v_z decreases with radius and $v_z(R) \approx v_z(0)\left(1 - \frac{2\nu}{3}\right)$. Some examples of this function are plotted in figure 1. Thus we have found our exact solution of the magnetohydrodynamic equation in the rigid rotation case. One notes that the increasing φ-pinch with increasing ν-value requires that $v_z(\rho)$ decreases with increasing ρ.

6. Constant axial velocity

If we assume that $v_z(\rho) = u_z =$ constant, within the cylinder, we get

$$\frac{F(\rho)}{\rho}\frac{\mathrm{d}F}{\mathrm{d}\rho} = \frac{1}{2}u_z^2\rho^2, \tag{35}$$

and use of this in the right-hand side of equation (30) gives the differential equation

$$A\left(\frac{\mathrm{d}G}{\mathrm{d}\rho}\right)^2 + [G(R) - G(\rho)]\,\rho\frac{\mathrm{d}G}{\mathrm{d}\rho} = \frac{1}{2}u_z^2\rho^2 \tag{36}$$

for G and $v_\varphi = \rho\omega(\rho) = \mathrm{d}G/\mathrm{d}\rho$. The initial condition is $G(0) = 0$.

Analytical solution is more difficult in this case because of the need to know $G(R)$ in order for the differential equation to be explicit. The differential equation can be rewritten as

$$R^2\omega^2(\rho) + 2\nu[G(R) - G(\rho)]\omega(\rho) = \nu u_z^2. \tag{37}$$

If we put $\rho = R$ here we get the exact expression,

$$R\omega(R) = \sqrt{\nu}u_z, \tag{38}$$

for the surface angular velocity. For small ν the angular velocity should vary slowly for consistency with the result of the previous section. Assuming $\omega(\rho) \approx \omega(0) =$ constant, one obtains $G(\rho) \approx \rho^2\omega(0)/2$, from (29). Use of this in (37) gives that

$$R\omega(0) \approx \sqrt{\frac{\nu}{1+\nu}}u_z, \tag{39}$$

at $\rho = 0$. Thus $\omega(\rho)$ increases with ρ and $\omega(R) \approx \sqrt{1+\nu}\,\omega(0)$.

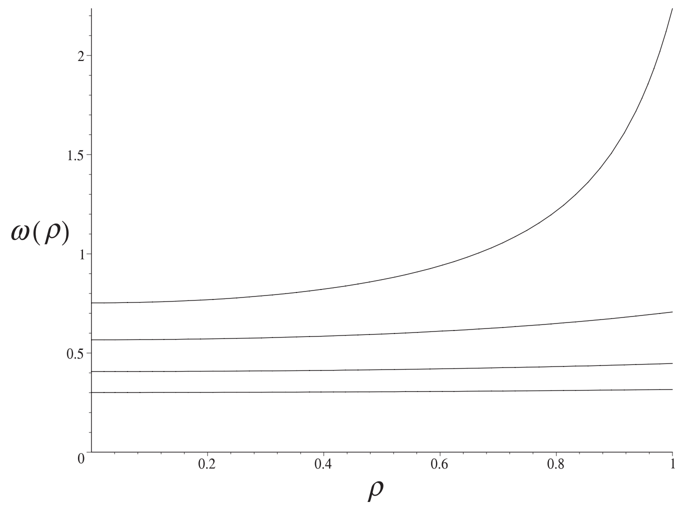

Figure 2. Plot of the function $\omega(\rho)$ of equation (45) for $u_z = R = 1$ and the same ν-values as in figure 1, that is, $\nu = 1/10, 1/5, 1/2, 5$, ordered from bottom and up.

We now ignore the problem with the $G(R)$-value and simply try to solve the differential equation

$$A\left(\frac{\mathrm{d}f}{\mathrm{d}\rho}\right)^2 + [B - f(\rho)]\rho\frac{\mathrm{d}f}{\mathrm{d}\rho} = \frac{1}{2}\rho^2. \tag{40}$$

for $f = G/u_z$, with $B = G(R)/u_z$. Maple 8 [26, 27] produces a set of four output solutions. Only two of these are zero at $\rho = 0$. Of these only one is increasing with ρ and clearly the one that interests us here. It can be simplified to give

$$G(\rho) = u_z\left[B + \sqrt{\frac{A}{2}}\frac{\left\{1 - W\left(2A\exp\left[\frac{C-\rho^2}{A}\right]\right)\right\}}{\sqrt{W\left(2A\exp\left[\frac{C-\rho^2}{A}\right]\right)}}\right] \tag{41}$$

where

$$C \equiv B^2 + A - A\ln 2 + A\ln\left(A + B^2 + B\sqrt{B^2+2A}\right) - 2A\ln A + B\sqrt{B^2+2A}, \tag{42}$$

and W is Lambert's W-function [28–30] defined in (12). Putting $u_z = R = 1$ one can solve for $B = G(1)$ and find

$$B = G(1) = \frac{1}{2}\frac{W(\exp[1+2\nu]) - 1}{\sqrt{\nu W(\exp[1+2\nu])}}. \tag{43}$$

Since $A = 1/2\nu$ the constant C is then known and (41) is an explicit solution with ν as parameter.

The derivative of W is [30]

$$\frac{\mathrm{d}W}{\mathrm{d}x} = \frac{1}{[1+W(x)]\exp[W(x)]} = \frac{W(x)}{x[1+W(x)]}. \tag{44}$$

Using this and the fact that $\mathrm{d}G/\mathrm{d}\rho = v_\varphi(\rho) = \rho\omega(\rho)$ we find that the angular velocity, for $u_z = R = 1$, is given by

$$\omega(\rho) = \sqrt{\frac{\nu}{W(\exp[1+2\nu-2\nu\rho^2])}}, \tag{45}$$

as a function of radius ρ. Some examples of this function for a selection of ν-values are plotted in figure 2.

We found above that when the magnetic field due to the angular velocity can be neglected a constant ω balances the z-pinch for constant v_z. When the magnetic field of the rotation becomes non-negligible for larger ν-values a φ-pinch develops and since we here demand that $v_z(\rho)$ be constant $(= u_z)$ it has to be balanced by an increasing angular velocity. This explains qualitatively the results of figure 2.

7. Discussion and conclusions

How must a cylindrical current distribution rotate in order to balance the pinching effect of its own magnetic field? That is the simple basic question that is answered by this work. Analytical solution was possible in all the specific cases treated above, but for the case of a constant translational velocity in the cylinder it required computers as well as some skill. The relevance of these idealized solutions for real plasmas can, of course, be questioned. In general, pressure as well as rotation will play a role. One reason for investigating these solutions is the well-known instability when pressure alone is balancing the pinching.

Concerning stability of our solutions, however, we can only offer the following heuristic and intuitive observations. The assumption of a constant density solution makes the pressure term vanish, but deformations of the cylinder will inevitably change the density and pressure will no longer be constant. Any changes in the total density will only cause acoustic oscillations about the equilibrium density. Any changes of the density of one species of charged particles will cause plasma oscillations about the equilibrium zero charge density. The solution is probably also stable against small random changes of the current density since such fluctuations are prevented from growing by the magnetic field. Nijboer *et al* [10] have investigated the stability properties of a related system.

The cited experimental, numerical and theoretical work also indicates that our class of solutions should possess some stability. After all, the solution consists of a bunch of charged particle trajectories passing self-consistently through their own magnetic field. If the solutions are stable enough to survive deformation they may in fact be precursors, so to speak, of the stable flux tubes of Faddeev and Niemi [21], which they identify with the filaments that plasmas frequently exhibit.

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